October Math Gems

PROBLEM OF THE WEEK 30

§1 Problems

Problem 1.1. Of 450 students assembled for a concert, 40 percent were boys. After a bus containing an equal number of boys and girls brought more students to the concert, 41 percent of the students at the concert were boys. Find the number of students on the bus.

Problem 1.2. Let a be a positive real number such that

$$4a^2 + \frac{1}{a^2} = 117.$$

Find

$$8a^3 + \frac{1}{a^3}.$$

Problem 1.3. Find $x^6 + \frac{1}{x^6}$ if $x + \frac{1}{x} = 3$.

Problem 1.4. Let $a_1 = 2021$ and for $n \geq 1$ let $a_{n+1} = \sqrt{4 + a_n}$. Then a_5 can be written as

$$\sqrt{\frac{m+\sqrt{n}}{2}} + \sqrt{\frac{m-\sqrt{n}}{2}}$$

where m and n are positive integers. Find $10m + n$.

Problem 1.5. The product

$$\left(\frac{1+1}{1^2+1} + \frac{1}{4}\right)\left(\frac{2+1}{2^2+1} + \frac{1}{4}\right)\left(\frac{3+1}{3^2+1} + \frac{1}{4}\right)\cdots\left(\frac{2022+1}{2022^2+1} + \frac{1}{4}\right)$$

can be written as $\frac{q}{2^r \cdot s}$, where r is a positive integer, and q and s are relatively prime odd positive integers. Find s.

Problem 1.6. Let a and b be positive real numbers satisfying

$$\frac{a}{b}\left(\frac{a}{b} + 2\right) + \frac{b}{a}\left(\frac{b}{a} + 2\right) = 2022.$$

Find the positive integer n such that

$$\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} = \sqrt{n}$$

Problem 1.7. Let a be a real number such that

$$5\sin^4\left(\frac{a}{2}\right) + 12\cos a = 5\cos^4\left(\frac{a}{2}\right) + 12\sin a.$$

There are relatively prime positive integers m and n such that $\tan a = \frac{m}{n}$. Find $10m + n$.

Problem 1.8. The sum of the solutions to the equation

$$x^{\log_2 x} = \frac{64}{x}$$

can be written as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 1.9. Let α, β, and γ denote the angles of a triangle. Show that

$$\sin\alpha + \sin\beta + \sin\gamma = 4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2},$$
$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4\sin\alpha\sin\beta\sin\gamma$$

and $\sin 4\alpha + \sin 4\beta + \sin 4\gamma = -4\sin 2\alpha \sin 2\beta \sin 2\gamma$.

Problem 1.10. Prove that no number in the sequence

$$11, 111, 1111, 11111, \ldots.$$

is the square of an integer.

Problem 1.11. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.

Problem 1.12. Prove the proposition: If a side of a triangle is less than the average (arithmetic mean) of the two other sides, the opposite angle is less than the average of the two other angles.

Problem 1.13. Prove that the only solution of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

in integers x, y, and z is $x = y = z = 0$.

Problem 1.14. Solve the following system of three equations for the unknowns x, y and z :

$$5732x + 2134y + 2134z = 7866$$
$$2134x + 5732y + 2134z = 670$$
$$2134x + 2134y + 5732z = 11464$$

Problem 1.15. A pyramid is called "regular" if its base is a regular polygon and the foot of its altitude is the center of its base. A regular pyramid has a hexagonal base the area of which is one quarter of the total surface-area S of the pyramid. The altitude of the pyramid is h. Express S in terms of h.

Problem 1.16. Show that each number of the sequence

$$49, 4489, 444889, 44448889, \ldots$$

is a perfect square.(Recall the formula for the sum of a geometric progression)

Problem 1.17. Solve the system

$$2x^2 - 4xy + 3y^2 = 36$$
$$3x^2 - 4xy + 2y^2 = 36$$

(One solution is easy to guess, but you are required to find all solutions. Knowledge of analytic geometry is not needed to solve this problem, but may help to understand the result-how?)

Problem 1.18. Solve the following system of three equations for the unknowns x, y, and z (a, b, and c are given) :

$$
\begin{aligned}
x^2y^2 + x^2z^2 &= axyz, \\
y^2z^2 + y^2x^2 &= bxyz, \\
z^2x^2 + z^2y^2 &= cxyz.
\end{aligned}
$$

Problem 1.19. Prove: If n is an integer greater than $1, n^{n-1} - 1$ is divisible by $(n-1)^2$.

Problem 1.20. Ten people are sitting around a round table. The sum of ten dollars is to be distributed among them according to the rule that each person receives one half of the sum that his two neighbors receive jointly. Is there just one way to distribute the money? Prove your answer.